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Qualitative Korovkin-Type Results on Conservative Approximation*

F. J. Muñoz-Delgado

Departamento de Matemáticas, Universidad de Jaén, 23071, Jaén, Spain
E-mail: fdelgado@ujaen.es

V. Ramírez-González

Departamento de Matemática Aplicada, Universidad de Granada, 18071, Granada, Spain

and

D. Cárdenas-Morales

Departamento de Matemáticas, Universidad de Jaén, 23071, Jaén, Spain

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In this paper, we present a generalization of the classical Korovkin theorem on positive linear operators. We deduce some convergence results for linear operators defined on $C^k[0, 1]$, that preserve some cones of functions related to shape properties. Finally, we show some examples. © 1998 Academic Press

1. INTRODUCTION AND NOTATIONS

The well-known result of Korovkin [4] states that for a sequence of positive linear operators $\{K_n\}_{n \geqslant 1}$, such that $K_n f$ converges uniformly to f in the particular cases $f(t) = 1$, $f(t) = t$, and $f(t) = t^2$, then it also converges for every continuous real function f. The set $\{1, t, t^2\}$ is called a Korovkin set. This result was a starting point for the development of a related theory. Since then many papers have appeared studying qualitative and quantitative Korovkin-type results for sequences of operators defined in many different spaces.

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On the other hand, interest in conservative approximation has increased. In this field the problem is to assign to each function another one belonging to a more reduced set, in such a way that if the function to be approximated verifies some shape properties then the approximating function also satisfies these properties. If the process is linear and positive, then Korovkin theorem and their first extensions give simple and nice methods to show the convergence. But if the process is not positive, then the study is more difficult, and these extensions usually do not give in practice suitable conditions for convergence.

In this direction, we state in advance the following propositions which, although they are very particular cases of the results we are proving in this paper, they help to understand our goal with this work. These results will extend in many aspects the mentioned Korovkin theorem.

In the sequel, we shall consider the terms positive, increasing, concave, and convex in a non-strict sense. Moreover, we use the notation $C^k[0, 1]$, $k \geqslant 0$, for the space of all real-valued and k-times continuously differentiable functions on $[0,1]$ endowed with the sup-norm $\|\cdot\|$, D^j for the jth differential operator, and $\mathbb{P}_k$ for the space of functions spanned by $\{e_0, e_1, ..., e_k\}$, where $e_i(x) = x^i$.

Proposition 1. *Let* $\{K_n\}_{n \geqslant 1}$, $K_n : C^2[0, 1] \to C^2[0, 1]$, *be a sequence of linear operators that map positive and convex functions onto positive functions.*

If $\|K_n e_j - e_j\| \to 0$ *as* $n \to \infty$ *for* $j = 0, 1, 2,$ *then*

$$\|K_n f - f\| \to 0 \qquad \text{as} \quad n \to \infty \qquad \text{for all} \quad f \in C^2[0, 1].$$

Remark. An analogous result cannot be stated for sequences of operators defined on $C^2[0, 1]$ that map positive and concave functions onto positive functions. To see this, it is sufficient to consider a polynomial operator $K : C^2[0, 1] \to \mathbb{P}_3$ defined in such a way that if $f \in C^2[0, 1]$ then $D^2(Kf)(x) = D^2 f(0) + (D^2 f(1) - D^2 f(0))\, x$ for $x \in [0, 1]$, $Kf(0) = f(0)$, and $Kf(1) = f(1)$. The constant sequence of linear operators $\{K_n\}_{n \geqslant 1}$, $K_n = K\ \forall n \geqslant 1$, holds the space $\mathbb{P}_3$ fixed and maps positive and concave functions onto positive functions but the operators are polynomial.

Proposition 2. *Let* $\{K_n\}_{n \geqslant 1}$, $K_n : C^2[0, 1] \to C^2[0, 1]$, *be a sequence of linear operators that map positive and concave functions onto concave functions.*

If $\|D^2(K_n e_j) - D^2 e_j\| \to 0$ *as* $n \to \infty$ *for* $j = 0, 1, 2, 3, 4,$ *then*

$$\|D^2(K_n f) - D^2 f\| \to 0 \qquad \text{as} \quad n \to \infty \qquad \text{for all} \quad f \in C^2[0, 1].$$

PROPOSITION 3. *Let* $\{K_n\}_{n\geqslant 1}$, $K_n: C^1[0,1] \to C^1[0,1]$, *be a sequence of linear operators that map positive and increasing functions onto increasing functions.*

If $\|D(K_n e_j) - De_j\| \to 0$ *as* $n \to \infty$ *for* $j = 0, 1, 2, 3$, *then*

$$\|D(K_n f) - Df\| \to 0 \qquad \text{as} \quad n \to \infty \qquad \text{for all} \quad f \in C^1[0,1].$$

Here is an outline of the rest of the paper: in Section 2 we show general criteria to obtain results similar to previous ones for sequences of operators that preserve shape properties. These results, that will be proved in Section 4 together with some corollaries, provided the key idea to state the main theorem of this paper that appears in Section 3. Finally, in Section 5, we make use of that theorem to show the convergence of two concrete sequences of operators and present a particular Korovkin-type theorem.

2. CONSERVATIVE APPROXIMATION IN $C^k[0,1]$

A set of functions C is called a cone if for every $f \in C$ and $\alpha \geqslant 0$, $\alpha f \in C$. In this section we define two different types of cones related to shape properties in the space $C^k[0,1]$ and we state two theorems for them.

Let $\sigma = \{\sigma_i\}_{i \geqslant 0}$ be a sequence with $\sigma_i \in \{-1, 0, 1\}$ and let h, k be two integers with $0 \leqslant h < k$ and $\sigma_h \sigma_k \neq 0$. We denote

$$C_{h,k}(\sigma) = \{f \in C^k[0,1] : \sigma_i D^i f \geqslant 0, h \leqslant i \leqslant k\}.$$

Let $\Gamma = \{i : h \leqslant i < k, \sigma_i \neq 0, \sigma_{i+1} = 0 \text{ and } \sigma_i \sigma_{i+2} \neq -1\}$.

If $\Gamma = \varnothing$ then we call $C_{h,k}(\sigma)$ a cone of type I.

If $\Gamma \neq \varnothing$ then we call $C_{h,k}(\sigma)$ a cone of type II.

We denote $\sigma^{[j]} = \{\sigma_i^{[j]}\}_{i \geqslant 0}$ with $\sigma_i^{[j]} = 0$ for $i \neq j$ and $\sigma_j^{[j]} = \sigma_j$.

THEOREM 1. *Let* $C_{h,k}(\sigma)$ *be a cone of type I or II and let* $\{K_n\}_{n \geqslant 1}$, $K_n: C^k[0,1] \to C^k[0,1]$, *be a sequence of linear operators.*

If $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma^{[k]})$ *and* $\|D^k(K_n e_j) - D^k e_j\| \to 0$ *as* $n \to \infty$ *for every* $j = h, ..., k+2$, *then*

$$\|D^k(K_n f) - D^k f\| \to 0 \qquad \text{as} \quad n \to \infty \qquad \text{for all} \quad f \in C^k[0,1].$$

THEOREM 2. *Let* $C_{h,k}(\sigma)$ *be a cone of type II, let* $r \in \Gamma$, *and let* $\{K_n\}_{n \geqslant 1}$, $K_n: C^k[0,1] \to C^k[0,1]$ *be a sequence of linear operators.*

If $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma^{[r]})$ *and* $\|D^r(K_n e_j) - D^r e_j\| \to 0$ *as* $n \to \infty$ *for every* $j = h, ..., k$, *then*

$$\|D^r(K_n f) - D^r f\| \to 0 \qquad \text{as} \quad n \to \infty \qquad \text{for all} \quad f \in C^k[0,1].$$

Note that Proposition 1 is a particular case of Theorem 2 taking $h=0$, $k=2$, and $\sigma=\{1, 0, 1, ...\}$. Also, Propositions 2 and 3 are particular cases of Theorem 1 taking, respectively, $h=0$, $k=2$, $\sigma=\{1, 0, -1, ...\}$ and $h=0$, $k=1$, $\sigma=\{1, 1, ...\}$.

Some results in this direction have appeared in previous papers: in [1], Altomare and Rasa, by means of envelope techniques, consider the case $\sigma=\{1, ..., 1, 0, 0, ...\}$; in [2], Brosowski, using the Dedekind-completion of a partially ordered vector space, considers the case $\sigma=\{1, 1, 0, 0, ...\}$; and in [3], Knoop and Pottinger work with the hypothesis $\sigma_i \in \{0, 1\}$.

3. AN EXTENSION OF KOROVKIN THEOREM

In this section we state a theorem that extends the previous results in the following aspects: the domain of the operators, the cones of functions, the role that played the D^j operator in Section 2, and consequently the subspaces in which the Korovkin sets must be located.

Let X be a compact subset of $\mathbb{R}^m$, $\mathbb{R}^X$ the space of all real-valued functions defined on X, and $C(X) \subset \mathbb{R}^X$ the subspace of all continuous functions. Let $B \subset \mathbb{R}^X$, let A be a subspace of $C(X)$ with $A \subseteq B$, and let L, $L: B \to \mathbb{R}^X$, be a linear operator satisfying $L(A) \subset C(X)$.

THEOREM 3. *Let $P=\{f \in B : Lf \geqslant 0\}$ and let C be a cone of A. Let V be a subspace of A satisfying the following properties*:

(v1) *There exists $u \in V$ such that $Lu(x)=1$ for $x \in X$.*

(v2) *For every point $z \in X$, there exists $\varphi_z \in V \cap C$ such that*

(a) $L\varphi_z(z)=0<L\varphi_z(x)\ \forall x \in X \backslash \{z\}$,

(b) $\forall f \in A, \exists \alpha=\alpha(f)>0 / \beta \geqslant \alpha \Rightarrow \beta\varphi_z+f \in C$.

Let $\{K_n\}_{n \geqslant 1}$, $K_n : A \to B$, be a sequence of linear operators satisfying the following properties:

(k1) $K_n(P \cap C) \subset P$ *for* $n \geqslant 1$.

(k2) *For every $f \in V$, $L(K_n f)$ converges uniformly to Lf as $n \to \infty$.*

Under these conditions, for every $f \in A$, $L(K_n f)$ converges uniformly to Lf as $n \to \infty$.

Proof. Let f be any function of A. Due to the continuity of Lf and the compactness of X, there exists a constant $M>0$ such that

$$-M<Lf(x)-Lf(y)<M \qquad \forall x, y \in X. \tag{3.1}$$

Furthermore, if a point $z \in X$ and a number $\varepsilon > 0$ are fixed, then there exists $\delta = \delta(z) > 0$ such that if $x \in B(z, \delta) = \{x \in X : |z - x| < \delta\}$ then

$$-\frac{\varepsilon}{3} < Lf(x) - Lf(z) < \frac{\varepsilon}{3}. \tag{3.2}$$

According to assumption (v2), there is a function φ_z that verifies (a) and (b). Let M_δ be the minimum value of the function $L\varphi_z$ on $X \backslash B(z, \delta)$.

By using Eq. (3.1) and Eq. (3.2), it is verified that

$$-\frac{\varepsilon}{3} - L\varphi_z(x)\frac{M\beta}{M_\delta} < Lf(x) - Lf(z) < \frac{\varepsilon}{3} + L\varphi_z(x)\frac{M\beta}{M_\delta} \qquad \forall x \in X, \quad \forall \beta > 1. \tag{3.3}$$

According to assumption (b), by taking a sufficiently large β, we have, using (v1), that

$$\frac{M\beta}{M_\delta}\varphi_z + \frac{\varepsilon}{3}u + f - Lf(z)\,u \in C$$

and

$$\frac{M\beta}{M_\delta}\varphi_z + \frac{\varepsilon}{3}u - f + Lf(z)\,u \in C.$$

But, using Eq. (3.3), these functions belong to P as well. Then, from assumption (k1), for $n \geqslant 1$, the image by K_n of these functions are functions in P and it follows that

$$\begin{aligned} -\frac{\varepsilon}{3}L(K_n u)(x) &- \frac{M\beta}{M_\delta}L(K_n\varphi_z)(x) \\ &\leqslant L(K_n f)(x) - Lf(z)\,L(K_n u)(x) \\ &\leqslant \frac{\varepsilon}{3}L(K_n u)(x) + \frac{M\beta}{M_\delta}L(K_n\varphi_z)(x) \qquad \forall x \in X, \end{aligned}$$

or equivalently

$$L(K_n f)(x) \leqslant Lf(z)\,L(K_n u)(x) + \frac{\varepsilon}{3}L(K_n u)(x) + \frac{M\beta}{M_\delta}L(K_n\varphi_z)(x) \qquad \forall x \in X \tag{3.4a}$$

and

$$L(K_n f)(x) \geqslant -\frac{\varepsilon}{3} L(K_n u)(x) + Lf(z)\, L(K_n u)(x) - \frac{M\beta}{M_\delta} L(K_n \varphi_z)(x) \quad \forall x \in X. \tag{3.4b}$$

By using assumption (k2) and (v1), there exists a number $N(\varepsilon, z)$ such that if $n \geqslant N(\varepsilon, z)$ then

$$-\frac{\varepsilon/9}{\varepsilon/3 + |Lf(z)|} + 1 < L(K_n u)(x) < \frac{\varepsilon/9}{\varepsilon/3 + |Lf(z)|} + 1 \qquad \forall x \in X \tag{3.5}$$

and

$$-\frac{(\varepsilon/9)\, M_\delta}{M\beta} + L\varphi_z(x) < L(K_n \varphi_z)(x) < \frac{(\varepsilon/9)\, M_\delta}{M\beta} + L\varphi_z(x) \qquad \forall x \in X. \tag{3.6}$$

Then, by using Eq. (3.4a), Eq. (3.5), and Eq. (3.6), if $n \geqslant N(\varepsilon, z)$ then

$$\begin{aligned} L(K_n f)(x) & \\ &< \frac{|Lf(z)|\, (\varepsilon/9)}{\varepsilon/3 + |Lf(z)|} + Lf(z) + \frac{(\varepsilon/3)(\varepsilon/9)}{\varepsilon/3 + |Lf(z)|} + \frac{\varepsilon}{3} + \frac{M\beta}{M_\delta}\left(L\varphi_z(x) + \frac{(\varepsilon/9)\, M_\delta}{M\beta}\right) \\ &= \frac{\varepsilon}{9} + Lf(z) + \frac{\varepsilon}{3} + \frac{\varepsilon}{9} + \frac{M\beta}{M_\delta} L\varphi_z(x) \\ &= \frac{2\varepsilon}{9} + \frac{\varepsilon}{3} + Lf(z) + \frac{M\beta}{M_\delta} L\varphi_z(x) \qquad \forall x \in X, \end{aligned} \tag{3.7a}$$

and by using Eq. (3.4b), Eq. (3.5), and Eq. (3.6), we have that if $n \geqslant N(\varepsilon, z)$ then

$$\begin{aligned} L(K_n f)(x) & \\ &> -\frac{(\varepsilon/3)(\varepsilon/9)}{\varepsilon/3 + |Lf(z)|} - \frac{\varepsilon}{3} - \frac{|Lf(z)|\, (\varepsilon/9)}{\varepsilon/3 + |Lf(z)|} + Lf(z) \\ &\quad - \frac{M\beta}{M_\delta}\left(L\varphi_z(x) + \frac{(\varepsilon/9)\, M_\delta}{M\beta}\right) \\ &= -\frac{\varepsilon}{9} - \frac{\varepsilon}{3} + Lf(z) - \frac{\varepsilon}{9} - \frac{M\beta}{M_\delta} L\varphi_z(x) \\ &= -\frac{2\varepsilon}{9} - \frac{\varepsilon}{3} + Lf(z) - \frac{M\beta}{M_\delta} L\varphi_z(x) \qquad \forall x \in X. \end{aligned} \tag{3.7b}$$

On the other hand, due to the continuity of $L\varphi_z$, there exists $\delta_z > 0$ with $\delta_z < \delta = \delta(z)$ such that if $x \in B(z, \delta_z)$ then

$$L\varphi_z(x) < \frac{(\varepsilon/9)\, M_\delta}{M\beta}. \tag{3.8}$$

Consequently, using Eq. (3.7a), Eq. (3.7b), and Eq. (3.8), if $x \in B(z, \delta_z)$ and $n \geqslant N(\varepsilon, z)$ then

$$|L(K_n f)(x) - Lf(z)| < \frac{2\varepsilon}{3}. \tag{3.9}$$

In this way, by Eq. (3.2) and Eq. (3.9), we have proved that for all $\varepsilon > 0$ and $z \in X$, we can find a positive integer $N(\varepsilon, z)$ and $\delta_z > 0$ such that if $n \geqslant N(\varepsilon, z)$ and $x \in B(z, \delta_z)$ then

$$|L(K_n f)(x) - Lf(x)| < \varepsilon.$$

The family of open subsets of X, $\{B(z, \delta_z)\colon z \in X\}$ is an open covering of X. As X is compact, there exists a finite subset J of X such that $\{B(z, \delta_z) : z \in J\}$ is a finite subcovering of X. Now, if we choose $N = \max\{N(\varepsilon, z) : z \in J\}$, it easily follows the uniform convergence of the sequence $L(K_n f)$. Indeed, for any point $x \in X$, there exists $z \in J$ in such a way that $x \in B(z, \delta_z)$. Consequently if $n > N$, then $|L(K_n f)(x) - Lf(x)| < \varepsilon$. ∎

4. APPLICATIONS AND REMARKS

In the sequel, we denote $\sigma^{(j)} = \{\sigma_i^{(j)}\}$ with $\sigma_i^{(j)} = \sigma_i$ for $i \neq j$ and $\sigma_j^{(j)} = 0$. Recall that we denote $\sigma^{[j]} = \{\sigma_i^{[j]}\}$ with $\sigma_i^{[j]} = 0$ for $i \neq j$ and $\sigma_j^{[j]} = \sigma_j$.

4.1. *Proof of Theorem* 1. We shall apply Theorem 3 as follows. Let L be the operator $\sigma_k D^k$, $A = B = C^k[0, 1]$, $C = C_{h,k}(\sigma^{(k)})$, and $V = \langle e_h, e_{h+1}, ..., e_{k+2}\rangle$. Observe that $P = C_{h,k}(\sigma^{[k]})$ and $P \cap C = C_{h,k}(\sigma)$.

Besides, we define $u = (1/k!)\, \sigma_k e_k$ and for every $z \in [0, 1]$ we define $\varphi_z \in V$ such that $D^k\varphi_z(x) = \sigma_k(x - z)^2$ for $x \in [0, 1]$ and successively $D^i\varphi_z(0) = \sigma_i(1 + \beta_i)$ with $\beta_i \geqslant \|D^{i+1}\varphi_z\|$ for $i = k-1, k-2, ..., h$.

We see that the hypotheses of Theorem 3 are verified. Indeed, if $h \leqslant i \leqslant k-1$ with $\sigma_i \neq 0$, then $\sigma_i D^i\varphi_z(x) = \sigma_i D^i\varphi_z(0) + \sigma_i \int_0^x (D^{i+1}\varphi_z) \geqslant \sigma_i D^i\varphi_z(0) - \beta_i \geqslant 1$ for $x \in [0, 1]$. Therefore $\varphi_z \in C$ and φ_z verifies assumption (v2)(b). The rest of the hypotheses are easily checked. ∎

4.2. COROLLARY OF THEOREM 1. *Let the sequence* $\{K_n\}_{n \geqslant 1}$ *be as in Theorem* 1 *and let* $C_{h,k}(\sigma)$ *be a cone of type I. If the sequence satisfies the following properties*:

(a) $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma)$ *for* $n \geqslant 1$,

(b) $\|D^k(K_n e_j) - D^k e_j\| \to 0$ *as* $n \to \infty$ *for* $j = h, h+1, ..., k+2$,

(c) $D^i(K_n e_j)$ *converges pointwise to* $D^i e_j$ *as* $n \to \infty$ *for* $i = h, ..., k-1$ *and* $j = h, ..., k$,

then

$$\|D^i(K_n f) - D^i f\| \to 0 \text{ as } n \to \infty \text{ for all } f \in C^k[0, 1] \text{ and } i = h, h+1, ..., k.$$

Proof. By using the theorem, we obtain that $D^k(K_n f)$ converges uniformly to $D^k f$ for every $f \in C^k[0, 1]$. Now, we shall prove the convergence of the lower order derivatives at certain points (we shall only use (a) and (c)).

Let $f \in C^k[0, 1]$ and let $r \in \langle e_h, ..., e_{k-1} \rangle$ and $s \in \langle e_h, ..., e_k \rangle$ satisfying the following properties:

(i) $D^k s = \sigma_k M$ with $M = \|D^k f\|$,

(ii) for $i = h, ..., k-1$, whenever $\sigma_i \neq 0$,

if $\sigma_i \sigma_{i+1} = 1$ then $D^i r(0) = D^i f(0)$ and $D^i s(0) = 0$,

if $\sigma_i \sigma_{i+1} = -1$ then $D^i r(1) = D^i f(1)$ and $D^i s(1) = 0$,

if $\sigma_{i+1} = 0$ then for $\alpha = 0, 1$, $D^i r(\alpha) = D^i f(\alpha)$ and $D^i s(\alpha) = 0$.

We define $w_1 = s + r - f$ and $w_2 = f + s - r$. It is verified that $w_1, w_2 \in C_{h,k}(\sigma)$ because $\sigma_k D^k w_1, \sigma_k D^k w_2 \geqslant 0$ and by recurrence it follows that $\sigma_i D^i w_1, \sigma_i D^i w_2 \geqslant 0$ for $i = k-1, k-2, ..., h$. Indeed, assume that $\sigma_l D^l w_t \geqslant 0$ for $l \geqslant i+1$ and $t = 1, 2$, then,

if $\sigma_i = 0$ we have obviously that $\sigma_i D^i w_t \geqslant 0$ for $t = 1, 2$,

if $\sigma_i \sigma_{i+1} \neq 0$, calling $\alpha_i = (1 - \sigma_i \sigma_{i+1})/2$, we have that for $t = 1, 2$.

$$\sigma_i D^i w_t(x) = \sigma_i D^i w_t(\alpha_i) + \sigma_i \int_{\alpha_i}^{x} D^{i+1} w_t = \sigma_i \int_{\alpha_i}^{x} D^{i+1} w_t \geqslant 0 \qquad \forall x \in [0, 1],$$

because if $\sigma_i \sigma_{i+1} = 1$ then $\alpha_i = 0$ and $\sigma_i D^{i+1} w_t \geqslant 0$, and if $\sigma_i \sigma_{i+1} = -1$ then $\alpha_i = 1$ and $\sigma_i D^{i+1} w_t \leqslant 0$. Finally,

if $\sigma_{i+1} = 0$ then $\sigma_i D^i w_t \geqslant 0$ for $t = 1, 2$ because they are concave functions that vanish at the points 0 and 1. Indeed, $\sigma_i D^{i+2} w_t = \sigma_{i+2} \sigma_{i+2} \sigma_i D^{i+2} w_t = -\sigma_{i+2} D^{i+2} \leqslant 0$ ($C_{h,k}(\sigma)$ is a cone of type I, so $\sigma_i \sigma_{i+2} = -1$), and for $\alpha = 0, 1$,

$$\begin{aligned}\sigma_i(D^i w_1)(\alpha) &= \sigma_i(D^i s)(\alpha) + \sigma_i(D^i r)(\alpha) - \sigma_i(D^i f)(\alpha)\\ &= 0 + \sigma_i(D^i f)(\alpha) - \sigma_i(D^i f)(\alpha) = 0,\\ \sigma_i(D^i w_2)(\alpha) &= \sigma_i(D^i f)(\alpha) + \sigma_i(D^i s)(\alpha) - \sigma_i(D^i r)(\alpha)\\ &= \sigma_i(D^i f)(\alpha) + 0 - \sigma_i(D^i f)(\alpha) = 0.\end{aligned}$$

By using assumption (a), we have that for $n \geqslant 1$, $\sigma_i D^i(K_n w_t) \geqslant 0$ for $t = 1, 2$ and $i = h, ..., k$.

In consequence, whenever $\sigma_i \neq 0$,

if $\sigma_i \sigma_{i+1} \neq 0$ then for $\alpha_i = (1 - \sigma_i \sigma_{i+1})/2$ we obtain

$$\sigma_i D^i(K_n w_1)(\alpha_i) = \sigma_i(D^i(K_n s)(\alpha_i) + D^i(K_n r)(\alpha_i) - D^i(K_n f)(\alpha_i)) \geqslant 0$$

and

$$\sigma_i D^i(K_n w_2)(\alpha_i) = \sigma_i(D^i(K_n f)(\alpha_i) + D^i(K_n s)(\alpha_i) - D^i(K_n r)(\alpha_i)) \geqslant 0.$$

Therefore $\sigma_i D^i(K_n s)(\alpha_i) \geqslant \sigma_i D^i(K_n f)(\alpha_i) - \sigma_i D^i(K_n r)(\alpha_i) \geqslant -\sigma_i D^i(K_n s)(\alpha_i)$, from which, using assumption (c), $(D^i s)(\alpha_i) = 0$, and $(D^i r)(\alpha_i) = (D^i f)(\alpha_i)$, it can be deduced that $D^i(K_n f)(\alpha_i) \to D^i f(\alpha_i)$. By analogy,

if $\sigma_{i+1} = 0$ then $D^i(K_n f)(0) \to D^i f(0)$ and $D^i(K_n f)(1) \to D^i f(1)$.

Now we show the uniform convergence of the lower order derivatives. If $\sigma_i \sigma_{i+1} \neq 0$, the uniform convergence of $D^i(K_n f)$ is a direct consequence of the uniform convergence of $D^{i+1}(K_n f)$ and the convergence of $D^i(K_n f)$ at a certain point. If $\sigma_{i+1} = 0$ we obtain the uniform convergence of $D^i(K_n f)$ and $D^{i+1}(K_n f)$ using the uniform convergence of $D^{i+2}(K_n f)$ and the convergence of $D^i(K_n f)$ at the points 0 and 1. ∎

Remark. In Theorem 1 and its Corollary the uniform convergence of $D^i(K_n e_j)$ to $D^i e_j$ for $i = h, ..., k$ and $j = h, ..., k+1$ is not a sufficient condition. To see this, a linear polynomial operator preserving the cone $C_{h,k}(\sigma)$ (type I) and fixing e_j for $j = h, ..., k+1$ can be constructed (see [5, 6]).

In fact, let $K: C^k[0, 1] \to \mathbb{P}_{k+1}$ be a linear operator such that

$$D^k(Kf)(x) = D^k f(0) + (D^k f(1) - D^k f(0))\, x \qquad \text{for} \quad x \in [0, 1],$$

and satisfying the following properties:

— for $i = h, ..., k-1$, whenever $\sigma_i \neq 0$,
  if $\sigma_{i+1} = 0$ then $D^i(Kf)(0) = D^i f(0)$ and $D^i(Kf)(1) = D^i f(1)$,
  if $\sigma_i \sigma_{i+1} \neq 0$ then $D^i(Kf)(\alpha_i) = D^i f(\alpha_i)$, with $\alpha_i = (1 - \sigma_i \sigma_{i+1})/2$,

— if $h > 0$ then for $i = h-1, ..., 0$, $D^i(Kf)(0) = D^i f(0)$.

The constant sequence of linear operators $\{K_n\}_{n\geqslant 1}$, $K_n = K$ $\forall n\geqslant 1$, preserves the cone $C_{h,k}(\sigma)$ and holds the space $\mathbb{P}_{k+1}$ fixed but the operators are polynomial.

4.3. *Proof of Theorem* 2. It is again a direct consequence of Theorem 3. By linearity, it is sufficient to consider $\sigma_r = 1$ (then $\sigma_{r+2} = 1$ or 0). Now, let L be the operator D^r, $A = B = C^k[0, 1]$, $C = C_{h,k}(\sigma^{(r)})$, and $V = \langle e_h, ..., e_k\rangle$. In this way $P = C_{h,k}(\sigma^{[r]})$ and $P \cap C = C_{h,k}(\sigma)$.

Besides, we define $u = (1/r!)\, e_r$ and for every $z \in [0, 1]$ we define $\varphi_z \in V$ such that $D^k\varphi_z(x) = \sigma_k$ for $x \in [0, 1]$ and if $k > r+3$, take successively $D^i\varphi_z(0) = \sigma_i(1+\beta_i)$ with $\beta_i \geqslant \|D^{i+1}\varphi_z\|$ for $i = k-1, k-2, ..., r+3$. Besides, $D^{r+2}\varphi_z(0) = 1+\beta_{r+2}$ with $\beta_{r+2} \geqslant \|D^{r+3}\varphi_z\|$. In this way, $D^{r+2}\varphi_z \geqslant 1$ and $D^r\varphi_z$ will be a strictly convex function that can be chosen in such a way that $D^r\varphi_z(z) = 0 < D^r\varphi_z(x)$ for $x \in [0, 1]\backslash\{z\}$ (to do this, take a function g such that $Dg = D^{r+1}\varphi_z$ and choose $D^r\varphi_z(x) = g(x) - g(z) - D^{r+1}\varphi_z(z)(x-z)$ for $x \in [0, 1]$). Finally, if $r > h$ we define $D^i\varphi_z(0) = \sigma_i(1+\beta_i)$ with $\beta_i \geqslant \|D^{i+1}\varphi_z\|$ for $i = r-1, r-2, ..., h$.

We have that $\varphi_z \in C$ and that φ_z verifies assumption (v2)(b) of Theorem 3 because $\sigma_i D^i\varphi_z \geqslant 1$ for $h \leqslant i \leqslant k$ such that $i \neq r$ and $\sigma_i \neq 0$. The rest of the hypotheses of this theorem are again easily checked. ∎

4.4. Corollary of Theorem 2. *Let the sequence $\{K_n\}_{n\geqslant 1}$ and the cone $C_{h,k}(\sigma)$ be as in Theorem 2 and let $l = \min\{\Gamma\}$.*

If $h < l$ and the sequence satisfies the following properties:

(a) *$K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma)$ for $n \geqslant 1$,*

(b) *$\|D^l(K_n e_j) - D^l e_j\| \to 0$ as $n \to \infty$ for $j = h, h+1, ..., k$,*

(c) *$D^i(K_n e_j)$ converges pointwise to $D^i e_j$ as $n \to \infty$ for $i = h, ..., l-1$ and $j = h, ..., k$,*

then

$$\|D^i(K_n f) - D^i f\| \to 0 \text{ as } n \to \infty \text{ for all } f \in C^k[0, 1] \text{ and } i = h, h+1, ..., l.$$

Proof. If $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma)$ then $K_n(C_{h,k}(\sigma)) \subset C_{h,k}(\sigma^{(l)})$ and by the theorem, we obtain that $D^l(K_n f)$ converges uniformly to $D^l f$ for all $f \in C^k[0, 1]$.

Now it suffices to prove the convergence of the lower order derivatives at certain points.

We define $\sigma^* = \{\sigma_i^*\}$ such that $\sigma_i^* = 1$ if $\sigma_i = 0$ and $l < i < k$, and $\sigma_i^* = \sigma_i$ otherwise. $C_{h,k}(\sigma^*)$ is a cone of type I and $C_{h,k}(\sigma^*) \subset C_{h,k}(\sigma)$.

As in the last corollary, we consider $f \in C^k[0,1]$ and we define $r \in \langle e_h, ..., e_{k-1} \rangle$ and $s \in \langle e_h, ..., e_k \rangle$ with the same conditions for $C_{h,k}(\sigma^*)$. It is verified that $w_1 = s + r - f$ and $w_2 = f + s - r$ belong to $C_{h,k}(\sigma^*)$, so for $n \geqslant 1$, $K_n w_1$, $K_n w_2 \in C_{h,k}(\sigma)$. By analogy, for $i = h, ..., l-1$ we obtain the following results:

(i) if $\sigma_i \sigma_{i+1} \neq 0$ then $D^i(K_n f)(\alpha_i) \to D^i f(\alpha_i)$ with $\alpha_i = (1 - \sigma_i \sigma_{i+1})/2$

(ii) if $\sigma_{i+1} = 0$ then $D^i(K_n f)(0) \to D^i f(0)$ and $D^i(K_n f)(1) \to D^i f(1)$.

Now again, the proof is ended with the convergence of the lower order derivatives at certain points. ∎

Remark. In Theorem 2 and its Corollary the uniform convergence of $D^i(K_n e_j)$ to $D^i e_j$ for $i = h, ..., l$ and $j = h, ..., k-1$ is not a sufficient condition. To see this, a linear polynomial operator preserving $C_{h,k}(\sigma)$ (type II) and fixing e_j for $j = h, ..., k-1$ can be constructed (see [5, 7]).

For example, if a natural number N is fixed, the classical Bernstein operator, B_N, in [0,1], represents a constant sequence of linear polynomial operators that fix $\mathbb{P}_1$ and preserve the cone $C_{0,2}(\sigma)$ with $\sigma = \{1, 0, 1, ...\}$.

5. EXAMPLES

Next some examples show how we can use Theorem 3, firstly to state different Korovkin-type results and secondly to prove the convergence of some particular sequences of operators.

EXAMPLE 1. *Let X be the closure of a bounded domain of $\mathbb{R}^m$, let Δ denote the Laplacian operator, and let $p_i(x) = x_i$ for all $x = (x_1, ..., x_m) \in X$ and $i = 1, ..., m$. Let $\{K_n\}_{n \geqslant 1}$, $K_n : C^2(X) \to \mathbb{R}^X$, be a sequence of linear operators satisfying that if $f \in C^2(X)$ with $f \geqslant 0$ and $\Delta f \geqslant 0$, then $K_n f \geqslant 0$.*

In these conditions $K_n f$ converges uniformly to $f \in C^2(X)$ as $n \to \infty$ if and only if the uniform convergence is satisfied for the functions $1, p_1, ..., p_m$, and $p_1^2 + \cdots + p_m^2$.

Proof. The result is a direct consequence of Theorem 3. To see this, let $A = C^2(x)$, $B = \mathbb{R}^X$, $Lf = f$, $C = \{f \in A : \Delta f \geqslant 0\}$, and $V = \langle 1, p_1, ..., p_m, p_1^2 + \cdots + p_m^2 \rangle$. Besides, we define $u(x) = 1$ and $\varphi_z(x) = (x_1 - z_1)^2 + \cdots + (x_m - z_m)^2$ for all $x \in X$ with $z = (z_1, ..., z_m)$ and $x = (x_1, ..., x_m)$. ∎

EXAMPLE 2 (Euler–Taylor Operators). *Let N be a natural number and let $H = \{f \in C^{N-1}[0,1] : \exists x_0 = 0 < x_1 < \cdots < x_m = 1$ such that $f \in C^N[x_i, x_{i+1}]$, $i = 0, ..., m-1\}$.*

Let $\{T_n\}_{n\geqslant 1}$, $T_n\colon C^N[0,1]\to H$, *be the sequence of operators defined by*

$$T_nf(x)=\begin{cases} f(0)+xDf(0)+\cdots+\dfrac{1}{N!}x^ND^Nf(0), \\ \qquad x\in\left[0,\dfrac{1}{n}\right] \\ \left(\displaystyle\sum_{s=0}^{N-1}\frac{1}{s!}\left(x-\frac{i}{n}\right)^s D^sT_nf\left(\frac{i}{n}\right)\right)+\dfrac{1}{N!}\left(x-\dfrac{i}{n}\right)^N D^Nf\left(\dfrac{i}{n}\right), \\ \qquad x\in\left(\dfrac{i}{n},\dfrac{i+1}{n}\right], \quad i=1,\dots,n-1. \end{cases}$$

Let J, $J: H\to\mathbb{R}^{[0,1]}$, *be the linear operator defined by*

$$Jf(x)=\frac{D^Nf_-(x)+D^Nf_+(x)}{2},$$

where D^Nf_- *and* D^Nf_+ *denote respectively the left and right derivatives of order* N *of* f.

Then $J(T_nf)$ *converges uniformly to* D^Nf *as* $n\to\infty$ *for all* $f\in C^N[0,1]$.

Proof. Euler–Taylor operators are not positive but they verify that if $f\in C^N[0,1]$, $f\geqslant 0$, $Df\geqslant 0$, ..., $D^{N-1}f\geqslant 0$, and $D^Nf\geqslant 0$, then $T_nf\geqslant 0$, $D(T_nf)\geqslant 0$, ..., $D^{N-1}(T_nf)\geqslant 0$, and $J(T_nf)\geqslant 0$. On the other hand, J verifies that $J(C^N[0,1])\subset C[0,1]$.

Now, the example follows if we apply Theorem 3 with $A=C^N[0,1]$, $B=H$, $L=J$, $C=\{f\in C^N[0,1]: f\geqslant 0, Df\geqslant 0, ..., D^{N-1}f\geqslant 0\}$, $V=\langle e_0, e_1, ..., e_{N+2}\rangle$, $u(x)=(1/N!)\,x^N$ for $x\in[0,1]$, and $\varphi_z\in V$ such that $L\varphi_z(x)=(x-z)^2$ and $D^i\varphi_z(x)>1$ for $x\in[0,1]$ and $i=0, 1, ..., N-1$. In this way, we obtain the result using that $T_ne_i=e_i$ for $i=0, ..., N$, $|J(T_ne_{N+1})-Je_{N+1})|\leqslant(N+1)!/n$ and $|J(T_ne_{N+2})-Je_{N+2})|\leqslant(N+2)!/n$.

Note that the uniform convergence of $D^i(T_nf)$ to D^if for $i=0, 1, ..., N-1$ is also obtained observing that $D^i(T_nf)(0)=D^if(0)$ for $i=0, 1, ..., N-1$. ∎

EXAMPLE 3. *Let* $f[x_0, x_1, ..., x_i]$ *denote the divided difference of the function* $f\in C[0,1]$ *in the points* $x_0, x_1, ..., x_i\in[0,1]$. *Let* $G=\{f\in C[0,1]: \exists M>0$ *such that* $|f[x_0,x_1,x_2]|\leqslant M\ \forall x_0,x_1,x_2\in[0,1]\}$ *and let* $\{K_n\}_{n\geqslant 1}$, $K_n: G\to C[0,1]$, *be the sequence of linear operators defined by*

$$K_n f(x) = \begin{cases} f(0) + f\left[0, \frac{1}{n}\right] x + f\left[0, \frac{1}{n}, \frac{2}{n}\right] x^2, \\ \qquad x \in \left[0, \frac{1}{n}\right] \\ K_n f\left(\frac{i}{n}\right) + DK_n f\left(\frac{i}{n}\right)\left(x - \frac{i}{n}\right) + f\left[\frac{i}{n}, \frac{i+1}{n}, \frac{i+2}{n}\right]\left(x - \frac{i}{n}\right)^2, \\ \qquad x \in \left(\frac{i}{n}, \frac{i+1}{n}\right], \qquad i = 1, ..., n-3 \\ K_n f\left(\frac{n-2}{n}\right) + DK_n f\left(\frac{n-2}{n}\right)\left(x - \frac{n-2}{n}\right) \\ \qquad + f\left[\frac{n-2}{n}, \frac{n-1}{n}, 1\right]\left(x - \frac{n-2}{n}\right)^2, \\ \qquad x \in \left(\frac{n-2}{n}, 1\right]. \end{cases}$$

Then $K_n f$ converges uniformly to f as $n \to \infty$ for all $f \in G$.

Proof. K_n are not positive operators (if $f(t) = t(1-t)$ then $K_n f(x) = x(1-x) - (x/n))$ but they verify that if $f \in G$, $f \geqslant 0$ and $f[x_0, x_1, x_2] \geqslant 0$ for all $x_0, x_1, x_2 \in [0, 1]$, then $K_n f \geqslant 0$.

We shall prove this statement through several steps. Firstly remember the formula for the divided differences we are using here.

$$f\left[\frac{i}{n}, \frac{i+1}{n}, \frac{i+2}{n}\right] = \frac{n^2}{2}\left(f\left(\frac{i+2}{n}\right) - 2f\left(\frac{i+1}{n}\right) + f\left(\frac{i}{n}\right)\right) \quad \text{for} \quad i = 0, 1, ..., n-2. \tag{5.1}$$

(1) Let f be a function in G. Then

$$D(K_n f)\left(\frac{i}{n}\right) = n\left(f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right)\right) \quad \text{for} \quad i = 0, 1, ..., n-2. \tag{5.2}$$

We shall prove these equalities by recurrence. Clearly $D(K_n f)(0) = n(f(1/n) - f(0))$. Now, assuming $D(K_n f)(i/n) = n(f((i+1)/n) - f(i/n))$ we have

$$
\begin{aligned}
D(K_n f)\left(\frac{i+1}{n}\right)
&= D(K_n f)\left(\frac{i}{n}\right) + 2\,\frac{n^2}{2}\left(\frac{i+1}{n} - \frac{i}{n}\right)\left(f\left(\frac{i+2}{n}\right) - 2f\left(\frac{i+1}{n}\right) + f\left(\frac{i}{n}\right)\right)\\
&= n\left(f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right) + f\left(\frac{i+2}{n}\right) - 2f\left(\frac{i+1}{n}\right) + f\left(\frac{i}{n}\right)\right)\\
&= n\left(f\left(\frac{i+2}{n}\right) - f\left(\frac{i+1}{n}\right)\right).
\end{aligned}
$$

If in addition we suppose that $f[x_0, x_1, x_1] \geqslant 0$ for all $x_0, x_1, x_2 \in [0, 1]$, it is also obtained by recurrence that

$$
K_n f\left(\frac{i}{n}\right) \geqslant f\left(\frac{i}{n}\right) \qquad \text{for} \quad i = 0, 1, ..., n-2. \tag{5.3}
$$

Indeed, $K_n f(0) = f(0)$ and if we assume that $K_n f(i/n) \geqslant f(i/n)$, we have, using (5.2), that

$$
\begin{aligned}
K_n f\left(\frac{i+1}{n}\right)
&= K_n f\left(\frac{i}{n}\right) + \left(\frac{1}{n}\right) D(K_n f)\left(\frac{i}{n}\right) + \frac{n^2}{2}\left(\frac{1}{n}\right)^2 \left(f\left(\frac{i+2}{n}\right)\right.\\
&\quad \left. - 2f\left(\frac{i+1}{n}\right) + f\left(\frac{i}{n}\right)\right)\\
&= K_n f\left(\frac{i}{n}\right) + \left(\frac{1}{n}\right) n\left(f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right)\right) + \frac{n^2}{2}\left(\frac{1}{n}\right)^2 \left(f\left(\frac{i+2}{n}\right)\right.\\
&\quad \left. - 2f\left(\frac{i+1}{n}\right) + f\left(\frac{i}{n}\right)\right)\\
&= K_n f\left(\frac{i}{n}\right) - \frac{1}{2} f\left(\frac{i}{n}\right) + \frac{1}{2} f\left(\frac{i+2}{n}\right)\\
&\geqslant f\left(\frac{i}{n}\right) - \frac{1}{2} f\left(\frac{i}{n}\right) + \frac{1}{2} f\left(\frac{i+2}{n}\right),
\end{aligned}
$$

but $\frac{1}{2}(f(i/n) + f((i+2)/n)) \geqslant f((i+1)/n)$ because $f[i/n, (i+1)/n, (i+2)/n] \geqslant 0$. This proves that $K_n f((i+1)/n) \geqslant f((i+1)/n)$.

(2) Let $f \in G$, $f \geqslant 0$, and $f[x_0, x_1, x_2] \geqslant 0$ $\forall x_0, x_1, x_2 \in [0, 1]$. Let $x \in [i/n, (i+1)/n]$ for any $i = 0, 1, ..., n-3$, then $K_n f(x) \geqslant 0$.

Indeed, Eqs. (5.1), (5.2), and (5.3) yield

$$K_n f(x) \geqslant f\left(\frac{i}{n}\right)+n\left(f\left(\frac{i+1}{n}\right)-f\left(\frac{i}{n}\right)\right)\left(x-\frac{i}{n}\right)$$
$$+\left(x-\frac{i}{n}\right)^2 f\left[\frac{i}{n}, \frac{i+1}{n}, \frac{i+2}{n}\right].$$

By using again the convexity condition on f we have that $f[i/n, (i+1)/n, (i+2)/n] \geqslant 0$ and $f(i/n)+n(f((i+1)/n)-f(i/n))(x-i/n) \geqslant f(x)$. This proves that $K_n f(x) \geqslant 0$.

(3) Let f as in (2) and let $x \in [(n-2)/n, 1]$. Then $K_n f(x) \geqslant 0$.

Firstly, we can write $K_n f(x) = K_n^1 f(x) + K_n^2 f(x)$ where

$$K_n^1 f(x) = K_n f\left(\frac{n-2}{n}\right)+\left(x-\frac{n-2}{n}\right) DK_n f\left(\frac{n-2}{n}\right)$$
$$+\frac{1}{2}\left(x-\frac{n-2}{n}\right)^2 f\left[\frac{n-2}{n}, \frac{n-1}{n}, 1\right]$$

and

$$K_n^2 f(x) = \frac{1}{2}\left(x-\frac{n-2}{n}\right)^2 f\left[\frac{n-2}{n}, \frac{n-1}{n}, 1\right].$$

Now, by using (5.1) and (5.2),

$$K_n^1 f(1) = K_n f\left(\frac{n-2}{n}\right)+2\left(f\left(\frac{n-1}{n}\right)-f\left(\frac{n-2}{n}\right)\right)$$
$$+\left(f(1)-2f\left(\frac{n-1}{n}\right)+f\left(\frac{n-2}{n}\right)\right)$$
$$= K_n f\left(\frac{n-2}{n}\right)+f(1)-f\left(\frac{n-2}{n}\right).$$

On the other hand, $D^2(K_n^1 f)(x) = f[(n-2)/n, (n-1)/n, 1]$. Consequently, if we define

$$g(x) = B_2 f(x) + K_n f\left(\frac{n-2}{n}\right)-f\left(\frac{n-2}{n}\right),$$

where $B_2 : C[(n-2)/n, 1] \to \mathbb{P}_2$ is the classical Bernstein operator in $[(n-2)/n, 1]$ and if we use that $g((n-2)/n) = K_n^1 f((n-2)/n)$, $g(1) = K_n^1 f(1)$, and $D^2 g(x) = D^2(K_n^1 f)(x)$, it is obtained that $K_n^1 f(x) = g(x)$.

Finally, $g(x)=K_n^1 f(x) \geqslant 0$ by the positivity of the Bernstein operator and Eq. (5.3) for $i=n-2$. Besides, $K_n^2 f(x) \geqslant 0$ because $f[(n-2)/n, (n-1)/n, 1] \geqslant 0$. This proves that $K_n f(x)=K_n^1 f(x)+K_n^2 f(x) \geqslant 0$.

Now, the proof of this example follows if we apply Theorem 3 with $A=G$, $B=C[0,1]$, $Lf=f$, $C=\{f \in A : f[x_0, x_1, x_2] \geqslant 0\ \forall x_0, x_1, x_2 \in [0,1]\}$, $V=\langle e_0, e_1, e_2 \rangle$, $u(x)=1$, and $\varphi_z(x)=(x-z)^2$ for $x \in [0,1]$. Note that we have already checked assumption (k1) of Theorem 3. Besides, assumption (v2)(b) is verified because if $f \in G$ then its second order divided differences are bounded. Finally $K_n e_i$ converges uniformly to e_i for $i=0,1,2$. Indeed, $K_n e_0=e_0$, $K_n e_1=e_1$, and $K_n e_2(x)=x/n+e_2(x)$ $\forall x \in [0,1]$ as it can be easily proved by recurrence. ∎

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